A few more remarks on symmetries

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Recall that a *symmetry* of a polyhedron Σ in three-space is an isometry f of space such that $f(\Sigma)=\Sigma$. During the revision session we have already argued that f is necessarily linear with respect to the vector space structure that uses the centre of mass as the origin. Hence, f is an orthogonal transformation. That is, if we fix a matrix A representing f with respect to an orthonormal basis, then $A^{-1}=A^T$. Now we will just choose such a basis and consider f as a matrix A. (Although the basis may change in various arguments.) We also observed that rotations are different from the reflections, because we can physically rotate the polyhedron onto itself, but a reflection isn't something we can implement. I would like to make a few more remarks about this distinction.

Recall that

$$\det(A)=\pm 1.$$

This is because $AA^T=I$, and hence,

$$1=\det(AA^T)=(\det(A))^2.$$

Let's first discuss the situation in the plane. In that case,

Lemma 1. $\det(A)=1$ *if and only if* A *is a rotation.*

I leave the proof as an exercise, which you've probably done already. On the other hand,

Lemma 2. $\det(A)=-1$ *if and only if* A *is a reflection.*

Proof. If A is a reflection, we can choose a basis $\{v,w\}$ so that v lies in the line of reflection and w is orthogonal to it. Thus $Av=v$ and $Aw=-w$. Therefore, the matrix of A with respect to the basis $\{v,w\}$ is

$$\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

and we get $\det(A)=-1$. The converse is a trick: Suppose $\det(A)=-1$. Then

$$\det(A+I)=\det(A+AA^T)=\det(A)\det(I+A^T)=-\det((I+A)^T)=-\det(I+A).$$

Therefore, $\det(A+I)=0$. So $A+I$ is non-invertible, and there is a vector v such that $Av=-v$. Let w be orthogonal to v. Since A preserves angles, Aw is also orthogonal to v, and hence, $Aw=\lambda w$. But then, $\det(A)=-\lambda$, so we must have $\lambda=1$ and $Aw=w$. Thus, A is the reflection across the line through w. $\square$

Now we move up to 3-space. Here as well,

Lemma 3. $\det(A) = 1$ *if and only if* A *is a rotation.*

Proof. We've discussed in tutorials the fact that a rotation has determinant 1. (Choose a basis with first element in the axis of rotation and the other two elements in the plane perpendicular to it, restricted to which the linear transformation will be a rotation.) On the other hand, if $\det(A) = 1$, then

$$\det(A - I) = \det(A - AA^T) = \det(A)\det(I - A^T) = \det(I - A) = -\det(A - I).$$

(We use here the odd dimension.) Therefore, $\det(A - I) = 0$, so there is a vector v such that $Av = v$. A must then preserve the plane perpendicular to v. Choose a basis $\{u, v\}$ for that plane. Then the matrix with respect to the basis $\{v, u, w\}$ will be of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix}$$

But then, the determinant of the $2\times$ 2 matrix is also 1, and so it's a rotation in the plane. From this and the fact that A is linear, one concludes that A is a rotation of three-space around the line through v. $\square$

If you feel uneasy about the phrase 'rotation around the line through v', you can just *define* a rotation to be a linear transformation that has the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(t) & -\sin(t) \\ 0 & \sin(t) & \cos(t) \end{pmatrix}$$

with respect to some basis, and then intuitively appreciate the fact that this correponds to your geometric sense of rotation.

I mentioned that *we* can carry out a reflection in the plane, even though a person living entirely in the plane could not. Here is one way to do it. Let R be a reflection in the plane represented as a matrix. Form in 3-space the matrix

$$A = \begin{bmatrix} \begin{matrix} & R & \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & -1 \end{bmatrix}$$

with respect to a basis consisting of two vectors in the plane and a normal vector. Now $\det(A) = 1$, so that it is a rotation that we can implement iin 3-space. Of course its effect on the plane we started out with is exactly the same as R. By choosing suitable bases to represent R, you can convince yourself that the rotation A is the most natural thing you would actually do, that is, a rotation through the angle π in the plane spanned by the vector that's flipped and the normal vector.

In 3-space that it's no longer true that $\det(A) = -1$ iff A is a reflection. For example, you can have a linear map with matrix

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & \cos(t) & -\sin(t) \\ 0 & \sin(t) & \cos(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(t) & -\sin(t) \\ 0 & \sin(t) & \cos(t) \end{bmatrix}$$

which is a rotation followed by a reflection, but not a reflection on its own. However, this is the general situation: Any matrix of determinant -1 is a rotation followed by a reflection. (Prove it!)

If we go up to four dimensional space, then it is no longer the case that $\det(A) = 1$ implies that A is a rotation. However,

Lemma 4. *In 4-space if* $\det(A) = 1$*, then A is a rotation or a composition of two rotations.*

Perhaps the meaning of a rotation in higher dimensional space is not so clear. Here is a definition that works in all dimensions: A linear transformation $L : V \longrightarrow V$ is a rotation if V can be written as an orthogonal direct sum $V = U \oplus W$ with U and W both preserved[1] by L and U two-dimensional, such that $L|U$ is a plane rotation while $L|W$ is the identity transformation. That is, a rotation is a rotation in a plane crossed with the identity in the complementary dimension. Expressed in terms of a matrix, this means there is a basis with respect to which the transformation is of the form

$$\begin{bmatrix} \cos(t) & -\sin(t) & 0 & 0 & \dots & 0 & 0 \\ \sin(t) & \cos(t) & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

In 4-space, you could imagine an alternative definition. That is, since we already know what a rotation in 3-space is, we could define a rotation in 4-space to be a tranformation that fixes one vector and is a rotation in the 3-space perpendicular to it. Fortunately, this definition is the same as the one above.

Proof. Let $p(x)$ be the characteristic polynomial of A. Since $p(x)$ is real, it can have 4,2, or 0 real roots. Any real root is ± 1. This is because if λ is an eigenvalue with eigenvector v, then $Av = \lambda v$ has to have the same length as v, so that $|\lambda| = 1$. I leave it to you to check that for any eigenvalue, the geometric multiplicity is equal to the algebraic multiplicity. This is of course false for arbitrary linear transformations, so you must use the orthogonality somehow.

[1] We say a subspace U of a vector space V is preserved by a linear transformation L if $L(U) \subset U$.

Suppose $p(x)$ has 4 real roots. Of course an even number of them are -1. Suppose they are all 1. Then A is the trivial rotation. Suppose two are -1. Then A has a matrix of the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Thus, A is a rotation parallel to the plane spanned by the last two basis vectors. If all four eigenvalues are -1, then

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is a composition of two rotations.

Suppose A has two real roots. Let U be its span of the corresponding eigenvectors and W its orthogonal complement. Both are preserved by A. (Why?) But $A|W$ has no real eigenvalues (again, why?), so it can't be a reflection (yet, again, why?). Hence, it's a rotation. Thus the eigenvalues on U are either both 1 or both -1. In the first case, A is a rotation parallel to one in W. In the latter case, it's a composition of $-I \oplus I_W$ and $I_U \oplus A|W$, two rotations. (You should decipher the meaning of this notation.)

Finally, suppose $p(x)$ has no real roots. Then we can write $p(x) = q(x)r(x)$ where q and r are quadratic real with no roots. We need the Cayley-Hamilton theorem, which we will prove next term:

$$p(A) = 0$$

as a linear map. Thus $q(A)r(A) = 0$. So either $q(A)$ or $r(A)$ will be non-invertible. Assume it is $q(A)$ without loss of generality. Write $q(x) = ax^2 + bx + c$ with $a \neq 0$. There is a non-zero vector v such that

$$aA^2v + bAv + cv = 0.$$

Using this, we see that the span U of v and Av are preserved by A. Furthermore, Av must be linearly independent from v, since otherwise, v would be an eigenvector. So U is two-dimensional. Again, let W be the orthogonal complement of U. Then $A|U$ and $A|W$ must both be rotations, and we have

$$A = (A|U \oplus I_W) \circ (I_U \oplus A|W),$$

a composition of two rotations.

$\square$

Using this lemma, one can show that a reflection in 3-space can be realized by a rotation or a composition of two rotations in 4-space. But of course, you can show this directly (exercise).