Review of *The Great Mathematical Problems* by Ian Stewart

Minhyong Kim

It is not uncommon among the general public for mathematical practice to be thought of mostly in terms of problem-solving, and the title of the book under review may reinforce this view. On the rare occasion that a mathematician is given some public attention, it is common enough for his or her work to be presented in connection with a famous problem or some sort, the impressiveness of the achievement being measured by the number of years the problem had stood open prior to its resolution.

It is indeed true that the solving of problems is an important part of mathematical activity. But this is also true of, say, theoretical physics. There as well, many problems, in fact of a mathematical nature, present themselves in day-to-day tasks. When it comes to working with actual numbers, that is to say, calculations, a physicist is probably much more proficient than a mathematician in solving all kinds of intricate problems. However, when the Nobel prize in physics is awarded, the media will most appropriately draw attention not to a problem solved, but to a *discovery*. What is perhaps not so well appreciated outside the expert community is that mathematics is also primarily about discovery, that is, of the shape and constituency of the mathematical universe, together with the intricate manner in which it interacts with the material world we live in.

As for the great problems, it seems fair to say that their primary use is to *efficiently generate the tools of discovery*. For a number of reasons, focusing on a problem of sufficient breadth and difficulty appears to aid greatly in generating the kind of mental energy that eventually manifests itself in the powerful tools that help to broaden and strengthen our grasp of the immense mathematical landscape. This is a time-honored tradition at least since the time of Archimedes, whose wide-ranging theoretical discoveries were somehow connected to the resolution of specific problems, often mechanical in nature. In a similar vein, among engineers, it seems to be rather well-known that the problem of constructing a rather specific device for a well-defined purpose is a great source of universal machines. One need only recall that Charles Babbage's 'analytical engine', designed to carry out simple arithmetical operations, is usually considered the first computer. Maybe I am just observing anew the old proverb that necessity is the mother of invention, together with the truism that specific needs of the right sort are actually very good at generating universal solutions.

Within mathematics proper, one of the best known problems solved in our times went by the name of Fermat's Last Theorem. But almost since its conception hundreds of years ago, the problem has been significant not so much for the elementary fact that it states, but because it ended up generating, maybe even accidentally, an enormous edifice of conceptual machinery that became permanently useful to the mathematical community. Most famous in this regard is the work of Ernst Kummer, which led to the development of *algebraic number theory*. We might also consider the fact that the statement of the theorem itself has had essentially no mathematical impact even after its resolution. However, the 'deformation theory' that Andrew Wiles developed for the proof has been the driving force behind many of the new advances in number theory in the last 15 years or so in connection with a grand theoretical framework known as the *the Langlands programme*. This was exactly the reason that Wiles was reluctant to follow up on his childhood dream and think about Fermat's problem, that is, until the work of Gerhard Frey, Kenneth Ribet, and others connecting it to mathematics of well-recognized depth and importance. He predicted that Fermat by itself would end up an isolated result. It was the connection to Langlands that gave him the excuse to really think it through with a degree of confidence that even partial results would contribute towards something global. The subsequent history, you could say, vindicated his intuitive sense of mathematical significance essentially precisely.

However, my real favorite example in this direction is the work of Évariste Galois, the 19th century French mathematician and revolutionary. His main concern, when he wasn't occupied with fomenting social unrest, was describing the roots to equations like

$$x^5 + x - 1 = 0.$$

With this modest goal in mind, he developed a mathematical structure known as a *group*. It was in the 20th century that the language of group theory became incorporated into relativity and quantum mechanics, and formed the basis of the classification of elementary particles, the ultimate constituents of matter, energy, and maybe even space itself.

*

If I have digressed in the paragraphs above, it was to prepare you for the observation that Ian Stewart's book as well is not primarily about mathematical problems, in spite of the title. It is in fact a highly illuminating survey of mathematical discovery, together with specific accounts of the thorny problems that spurred the explorers along over millenia, and will continue to do so for the foresseable future.

But the reference to thorns and spurs is also a bit ill-placed. The final resolution of a mathematical problem is usually stated as a *theorem*, a word you may have heard, which we used already in connection to Fermat. In the introduction to the book, Stewart quotes the British mathematician Christopher Zeeman, one of the originators of *catastrophe theory*:

> 'A mathematical theorem is an intellectual resting point. You can stop, get your breath back, and feel you've got somewhere definite.'

Zeeman's metaphor recapitulates well the historical perspective on the role of problems in humanity's mathematical journey. Those problems are good that end up serving successfully as highly visible signposts, which provide thereby the substantive short-term goals, the special points on the road at which you can rest for a moment or even a day in the course of a never-ending pilgrimage.

As you read through this capable survey of the most important problems in modern mathematics written by a very capable guide, I do hope you are able to get a concrete and genuine sense of this ongoing process, which is, after all, the real story directing the disparate strands of conjectures, theorems, and proofs presented therein. Best of all would be if some of the delightful chapters were to inspire you to join in yourself.