Orthogonal bases

Let F be a field of characteristic different from 2, for example, $\mathbb{Q}$ or $\mathbb{R}$ or $\mathbb{C}$. Let $< \cdot, \cdot >$ be a symmetric bilinear form on a finite-dimensional vector space V. There is thee question of finding an orthogonal basis for V with respect to the bilinear form. [Note that we are not necessarily searching for an *orthonormal* basis.]

Lemma 0.1. *If $< \cdot, \cdot >$ is not the zero form, then there is a vector v such that $< v, v > \neq 0$.*

Proof. Let v and w be vectors such that $< v, w > \neq 0$. If $< v, v > \neq 0$ or $< w, w > \neq 0$, then we are done. If they are both zero, then we see that

$$< v + w, v + w >= 2 < v, w > \neq 0.$$

□

One finds an orthogonal basis by induction on the dimension. If $< \cdot, \cdot >$ is the zero form, we are done: any basis is orthogonal. If not, let b_1 be such that $< b_1, b_1 > \neq 0$. Then we see easily that

$$V = [b_1] \oplus [b_1]^{\perp},$$

where $[b_1]$ refers to the subspace generated by b_1. Since $\dim[b_1]^{\perp} < \dim V$, we can apply induction.

Let's see how this works in practice for problem 5 in sheet 4. There, V is the vector space of 2×2 real matrices, and

$$< A, B >= Tr(AB).$$

We start with the standard basis

$$c_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad c_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$c_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad c_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

We can take $b_1 = c_1$, since $< b_1, b_1 >= 1$. Now, one checks easily that each of c_1, c_3, c_4 are orthogonal to b_1, so that

$$[b_1]^{\perp} = [c_2, c_3, c_4].$$

We see quickly that the form restricted to $[c_2, c_3, c_4]$ is non-zero. At this point, we shouldn't choose c_2 to be the second basis vector, since $< c_2, c_2 >= 0$. However, $< c_4, c_4 >= 1 \neq 0$. So we let $b_2 = c_4$. Now we could have computed $[b_2]^{\perp}$ inside $[c_2, c_3, c_4]$ by using the Gram-Schimdt process, but this would only have revealed that c_2 and c_3 are already both orthogonal to $b_2 = c_4$. Thus, $[b_2]^{\perp} = [c_2, c_3]$. Now, $< c_2, c_2 >=< c_3, c_3 >= 0$, but $< c_2, c_3 >= 1$. Thus, we can take $b_3 = c_2 + c_3$. Finally, one easily checks that $b_4 = c_2 - c_3$ is orthogonal to b_3. (Once again, we could have used Gram-Schmidt to come up with the orthogonal vector.)