Remark on the coefficients and roots of a polynomial

Let

$$f(x) = x^d + a_{d-1}x^{d-1} + a_{d_2}x^{d-2} + \cdots + a_1 x + a_0$$

be a monic polynomial of degree zero with coefficient in an algebraically closed field k. Clearly, the set of such polynomials can be identified with the affine d-space $\mathbf{A}_k^d$ over k by identifying such a polynomial simply with the coefficients $(a_{d-1}, \ldots, a_0)$. Suppose f is factorized into linear terms as

$$f(x) = (x - t_1)(x - t_2) \cdots (x - t_d).$$

Thus, the t_i will be the roots of the polynomial. Then it is elementary to check that the relationship between the roots and the coefficients is given by

$$a_i = (-1)^i s_i(t_1, t_2, \ldots, t_d).$$

where

$$s_i(t_1, \ldots, t_d) = \sum_{k_1 < k_2 < \cdots, k_i} t_{k_1} t_{k_2} \cdots t_{k_i}.$$

For example,

$$a_1 = -(t_1 + t_2 + \cdots + t_d), \quad a_d = (-1)^d t_1 t_2 \cdots t_d.$$

We conclude that the map

$$\pi : \mathbf{A}_k^d \to \mathbf{A}_k^d$$

given by

$$\pi(t_1, \ldots, t_d) = (-s_1(t_1, \ldots, t_d), s_2(t_1, \ldots, t_d), -s_3(t_1, \ldots, t_d), \ldots, (-1)^d s_d(t_1, \ldots, t_d))$$

is an algebraic map such that $\pi^{-1}(a_{d-1}, \ldots, a_0)$ is the set of orderings of the roots of the polynomial

$$x^d + a_{d-1}x^{d-1} + a_{d_2}x^{d-2} + \cdots + a_1 x + a_0.$$